Positivity results for Stanley's character polynomials

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Abstract

In Stanley [16], the author introduces expressions for the normalized characters of the symmetric group and states some positivity conjectures for these expressions. Here, we give an affirmative partial answer to Stanley's positivity conjectures about the expressions using results on Kerov polynomials. In particular, we use new positivity results in Goulden and Rattan [7]. We shall see that the generating series $C(t)$ introduced in [7] is critical to our discussion.

1 Introduction

A *partition* is a weakly ordered list of positive integers $\lambda = \lambda_1\lambda_2\dots\lambda_k$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. The integers $\lambda_1, \dots, \lambda_k$ are called the *parts* of the partition λ, and we denote the number of parts by $l(\lambda) = k$. If $\lambda_1 + \dots + \lambda_k = d$, then λ is a partition of d, and we write $\lambda \vdash d$. We denote by $\mathcal{P}$ the set of all partitions, including the single partition of 0 (which has no parts). For partitions $\omega, \lambda \vdash n$ let $\chi_\omega(\lambda)$ be the character of the irreducible representation of the symmetric group $\mathfrak{S}_n$ indexed by ω, and evaluated on the conjugacy class $\mathcal{C}_\lambda$ of $\mathfrak{S}_n$, which consists of all permutations whose disjoint cycle lengths are specified by the parts of λ.

Various scalings of irreducible symmetric group characters have been considered in the recent literature. The *central character* is given by

$$\widetilde{\chi}_\omega(\lambda) = |\mathcal{C}_\lambda| \frac{\chi_\omega(\lambda)}{\chi_\omega(1^n)},$$

where $\chi_\omega(1^n)$ is the *degree* of the irreducible representation indexed by ω. For results about the central character, see, for example, [4, 5, 10]. The scaling to be discussed in this paper, the *normalized character*, is given for any partitions $\omega \vdash n$ and $\mu \vdash k$, where $k \leq n$, by

$$\widehat{\chi}_\omega(\mu\, 1^{n-k}) = n(n-1)\cdots(n-k+1)\frac{\chi_\omega(\mu\, 1^{n-k})}{\chi_\omega(1^n)}.$$

For the conjugacy class $C_{k\ 1^{n-k}}$ only, the normalized character and the central character are related by the following:

$$\widehat{\chi}_\omega(k1^{n-k}) = n(n-1)\cdots(n-k+1)\frac{\chi_\omega(k1^{n-k})}{\chi_\omega(1^n)} = k\widetilde{\chi}_\omega(k1^{n-k}).$$

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The subject of this paper is a particular polynomial expression for the normalized character, introduced in Stanley [16]. Consider the partition with p_i parts of size q_i, for i from 1 to m, with q_1 the largest part. Thus, $p_1, p_2, \ldots, p_m$ are positive integers and $q_1 > q_2 > \cdots > q_m$ (see Figure 1). We denote this partition by $\mathbf{p} \times \mathbf{q}$. Define the series F_k in indeterminates

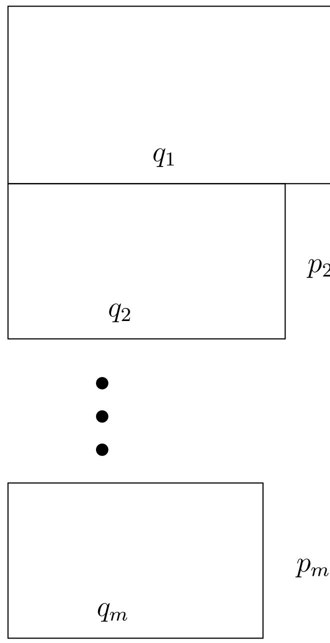

Figure 1: The shape $\mathbf{p} \times \mathbf{q}$.

$p_1, \ldots, p_m, q_1, \ldots, q_m$ by

$$F_k(p_1, p_2, \ldots, p_m;\ q_1, q_2, \ldots, q_m) = \widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(k\ 1^{n-k}) \tag{1}$$

We often denote $(p_1, \ldots, p_m)$ by $\mathbf{p}$ and $(q_1, \ldots, q_m)$ by $\mathbf{q}$, giving us the notation $F_k(\mathbf{p};\ \mathbf{q})$ for $F_k(p_1, p_2, \ldots, p_m;\ q_1, q_2, \ldots, q_m)$. The following theorem appears in Stanley [16, Proposition 1].

Theorem 1.1 (Stanley). *$F_k(\mathbf{p};\ \mathbf{q})$ is a polynomial in the p's and q's such that $F_k(1, 1, \ldots, 1;\ -1, -1, \ldots, -1) = (k + m - 1)_k$.*

In light of this theorem, we call the polynomials in (1) *Stanley's character polynomials*. These polynomials are the main objects in this paper. For example, for the case $m = 2$, the first two Stanley polynomials are

$$\begin{aligned} F_1(a, p;\ b, q) &= -ab - pq, \\ F_2(a, p;\ b, q) &= -a^2b + ab^2 - 2\,apq - p^2q + pq^2 \end{aligned}$$

where we have set $p_1 = a, p_2 = p, q_1 = b$ and $q_2 = q$.

1.1 Main results

In [16], Stanley generalizes $F_k(\mathbf{p};\ \mathbf{q})$ to

$$F_\mu(\mathbf{p};\ \mathbf{q}) = \widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(\mu\ 1^{n-k}),$$

where μ is a partition of k. Stanley states that $F_\mu(\mathbf{p};\ \mathbf{q})$ is, by the Murnaghan-Nakayama rule, a polynomial with integer coefficients. In [16, Conjecture 1], Stanley gives a positivity conjecture for a variant of the series $F_\mu(\mathbf{p};\ \mathbf{q})$. For convenience, we use the notation $-\mathbf{q} = (-q_1, -q_2, \ldots, -q_m)$, and $F_\mu(\mathbf{p};\ -\mathbf{q})$ is the series $F_\mu(\mathbf{p};\ \mathbf{q})$ with q_i replaced by $-q_i$. Call any series $T(\mathbf{p};\ \mathbf{q})$ in the indeterminates p's and q's $\mathbf{p},\mathbf{q}$-*positive* if the coefficients of all terms are positive.

Conjecture 1.2 (Stanley). *For any partition $\mu \vdash k$, the series $(-1)^k F_\mu(\mathbf{p}; -\mathbf{q})$ is $\mathbf{p},\mathbf{q}$-positive.*

Stanley only proves this in the case $m = 1$, the so-called rectangular case as in this case the shape $\mathbf{p} \times \mathbf{q}$ is the rectangle with p_1 parts all equal to q_1. We drop the subscript 1 in this case and say $p \times q$ is the partition with p parts all equal to q. In the rectangular case, Stanley proves positivity by giving a stronger result; he gives a combinatorial interpretation for the coefficients, given in [16, Theorem 1] and stated below.

Theorem 1.3 (Stanley). *Suppose that $p \times q \vdash n$ and $\mu \vdash k$ for $k \leq n$. Let λ_μ be any fixed permutation in the conjugacy class indexed by μ in $\mathfrak{S}_k$. Then,*

$$\widehat{\chi}_{p\times q}(\mu\, 1^{n-k}) = (-1)^k \sum_{\substack{u,\nu \\ u\nu=\lambda_\mu}} p^{\ell(u)}(-q)^{\ell(\nu)}.$$

For general m, Conjecture 1.2 remains open. In fact, there is no proof even in the case where μ has one part, that is, it is not yet known whether $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$ is $\mathbf{p},\mathbf{q}$-positive. This generating series will be the focus of this paper and we address its $\mathbf{p},\mathbf{q}$-positivity.

Stanley does state that the terms of highest degree, the terms of degree $k+1$, of $(-1)^k F_k(\mathbf{p};\ \mathbf{q})$ have a particularly nice form, and are given in (13) below. He does not, however, prove that these terms are $\mathbf{p},\mathbf{q}$-positive but does state that Elizalde has given a proof of this in private communication to him (see (14) below). The proof by Elizalde does not appear to be anywhere in the literature.

In this paper, we give a new proof of Theorem 1.3. We do this using *shift symmetric functions*. This new proof, we hope, is simpler and makes the result more transparent. Furthermore, it highlights the already known connection between shift symmetric functions and the normalized character. As for the general case of $F_k(\mathbf{p};\ \mathbf{q})$, we give a proof of $\mathbf{p},\mathbf{q}$-positivity of the terms of highest degree, the terms of degree $k+1$, in $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$ (as mentioned above, this was also proved by Elizalde, but his proof does not appear in the literature). We also give a proof that the terms of degree $k-1$ and $k-3$ in $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$, the terms of second and third highest degree, are also $\mathbf{p},\mathbf{q}$-positive, which are new results. We do this by using recent results concerning *Kerov's polynomials*. For Kerov's polynomials, there is a notion of R-positivity and a new notion of C-positivity introduced in Goulden and Rattan [7], which we shall use to show our positivity results for $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$. Finally, we end the paper by showing that C-positivity of Kerov's polynomials implies $\mathbf{p},\mathbf{q}$-positivity of $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$, also a new result.

The necessary results on Kerov's polynomials are reviewed in the next section.

1.2 Kerov polynomials

We adapt the following description from Biane [2, 3]: consider the Young diagram of ω, in the French convention (see [11, footnote page 2]), and translate it, if necessary, so that the bottom left of the diagram is placed at the origin of an (x, y) plane. Finally, rotate the diagram counter-clockwise by $45°$. Note that ω is uniquely determined by the curve $\tau_\omega(x)$ (see Figure 1.2). The value of $\tau_\omega(x)$ is equal to $|x|$ for large negative or positive values of

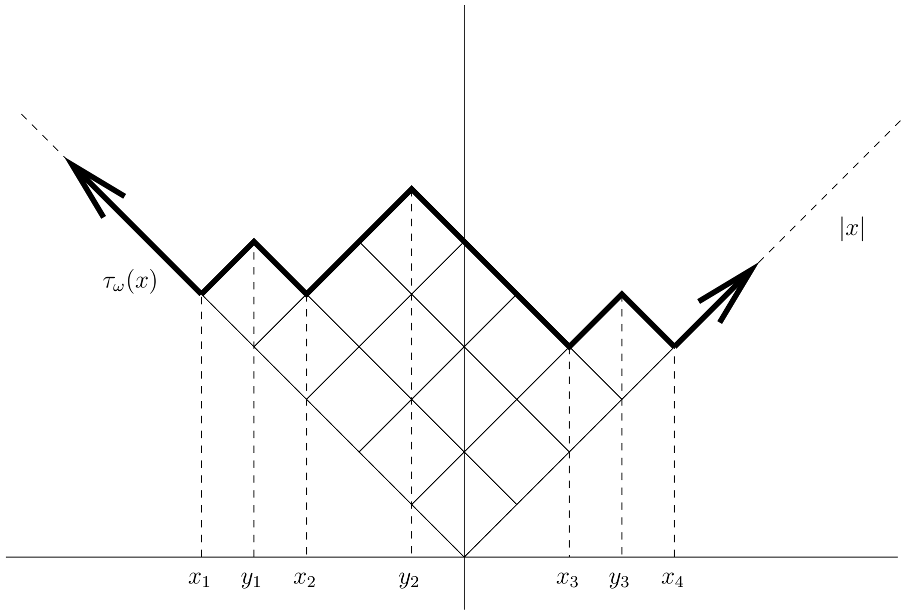

Figure 2: The partition (4 3 3 3 1) of 14, drawn in the French convention, and rotated by 45°.

x and it is clear that $\tau'_\omega(x) = \pm 1$, where differentiable. The points x_i and y_i are the x-coordinates of the local minima and maxima, respectively, of the curve $\tau_\omega(x)$. We suitably scale the size of the boxes in our Young diagram so that the points x_i and y_i are integers. Setting $\sigma_\omega(x) = (\tau_\omega(x) - |x|)/2$, consider the function

$$H_\omega(z) = \frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{z - x}\ \sigma'_\omega(x)\ dx.$$

Carrying out the above integration one obtains

$$H_\omega(z) = \frac{\prod_{i=1}^{m-1}(z - y_i)}{\prod_{i=1}^{m}(z - x_i)}, \tag{2}$$

where m is the number of nonempty rows in the Young diagram of ω. Now let $R_\omega(z) = \sum_{i\geq 0} R_i(\omega) z^i$ be defined by

$$R_\omega(z) = z H_\omega^{\langle -1 \rangle}(z), \tag{3}$$

where $\langle -1 \rangle$ denotes compositional inverse. We will be applying Lagrange inversion (see Goulden and Jackson [6, Section 1.2] or Stanley [15, Theorem 5.4.2]) to (2) and (3), in which case we change $H_\omega(z)$ to a formal power series. We then obtain

$$R_\omega(z) = \frac{z}{(H_\omega(1/z))^{\langle -1 \rangle}}. \tag{4}$$

Briefly, the origins of the series $H_\omega(z)$, and in fact Kerov's polynomials, come from attempting to answer asymptotic questions about the characters of the symmetric group using free probability. In that context $H_\omega(z)$ is called the *moment generating series* (traditionally denoted $G_\omega(z)$) and $R_\omega(z)$ is the *free cumulant generating series* (traditionally denoted $K_\omega(z)$). We refer the reader to Biane [1, 3] for the background on asymptotics of characters and free probability.

Finally, the polynomials we are concerned with involve the $R_i(\omega)$'s and are given in the following theorem. They first appeared in Biane [3, Theorem 1.1].

Theorem 1.4 (Biane). *For $k \geq 1$, there exist universal polynomials Σ_k, with integer coefficients, such that*

$$\widehat{\chi}_\omega(k\ 1^{n-k}) = \Sigma_k(R_2(\omega), R_3(\omega), \ldots, R_{k+1}(\omega)), \tag{5}$$

for all $\omega \vdash n$ with $n \geq k$.

Biane attributes Theorem 1.4 to Kerov, who described this result in a talk at an IHP conference in 2000, but a proof first appears in a later paper of Biane [1]. The polynomials Σ_k are known as *Kerov's character polynomials*. They are referred to as "universal polynomials" in Theorem 1.4 to emphasize that they are independent of ω and n, subject only to $n \geq k$. Thus we write them with $R_i(\omega)$ replaced by an indeterminate R_i, $i \geq 2$. In indeterminates R_i, the first six of Kerov's character polynomials, as listed in [3], are given below:

$$\begin{aligned}
\Sigma_1 &= R_2\\
\Sigma_2 &= R_3\\
\Sigma_3 &= R_4 + R_2\\
\Sigma_4 &= R_5 + 5R_3\\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2\\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3
\end{aligned} \tag{6}$$

We note that once one knows Kerov's polynomial Σ_k in order to find $\widehat{\chi}_\omega(k\ 1^{n-k})$ all one needs is to construct the series $H_\omega(z)$ as in (2) and apply Lagrange inversion to (4) to find the series $R_\omega(z)$, giving the $R_i(\omega)$. Substituting these values into the k^{th} polynomial will then give the normalized character $\widehat{\chi}_\omega(k\ 1^{n-k})$.

Note that all coefficients appearing in this list are positive. It is conjectured that this holds in general: that for any $k \geq 1$, all nonzero coefficients in Σ_k are positive (see Theorems 1.5, 1.6 and 1.7 for positivity results obtained so far for Kerov's polynomials). Kerov's polynomials remain somewhat of a mystery, in spite of recent efforts by Biane [1], Śniady [14] and Goulden and Rattan [7]. In particular, in [7] the authors introduce the polynomial $C(t) = \sum_{m\geq 0} C_m t^m$ given by

$$C(t) = \frac{1}{1 - \sum_{i\geq 2}(i-1)R_i t^i}. \tag{7}$$

From the definition of $C(t)$ we see the C_m are polynomials in the R_i's, with $C_0 = 1$, $C_1 = 0$, and

$$C_m = \sum_{\substack{j_2, j_3, \ldots \geq 0\\ 2j_2+3j_3+\ldots=m}} (j_2 + j_3 + \ldots)! \prod_{i\geq 2} \frac{((i-1)R_i)^{j_i}}{j_i!}, \quad m \geq 2. \tag{8}$$

Writing Kerov's polynomials in terms of the C's we have (from Goulden and Rattan [7, page 7])

$$\begin{aligned}
\Sigma_3 - R_4 &= C_2\\
\Sigma_4 - R_5 &= \tfrac{5}{2}C_3\\
\Sigma_5 - R_6 &= 5\,C_4 + 8\,C_2\\
\Sigma_6 - R_7 &= \tfrac{35}{4}\,C_5 + 42\,C_3\\
\Sigma_7 - R_8 &= 14\,C_6 + \tfrac{469}{3}\,C_4 + \tfrac{203}{3}\,{C_2}^2 + 180\,C_2\\
\Sigma_8 - R_9 &= 21\,C_7 + \tfrac{1869}{4}\,C_5 + \tfrac{819}{2}\,C_3C_2 + 1522\,C_3
\end{aligned} \tag{9}$$

We will, henceforth, call the expansions in (6) and (9) the *R-expansions* and *C-expansions*, respectively, of Kerov's polynomials. We will also call the property that all coefficients of the R's are positive *R-positivity* and *C-positivity* is analogously defined for $\Sigma_k - R_{k+1}$. It follows from (8) that C-positivity of Kerov's polynomials implies R-positivity.

Define the weight of a monomial $R_{i_1}^{j_1} R_{i_2}^{j_2} \cdots R_{i_m}^{j_m}$ to be $\sum_{t=1}^m i_t j_t$ (and analogously for monomials in C's). Further let $\Sigma_{k,2n}$ be the terms of weight $k+1-2n$ in Σ_k. From the combinatorial origins of Kerov's polynomials, it follows that in the k^{th} Kerov polynomial Σ_k the terms of weight $k \pmod 2$ each have zero coefficient (see Biane [1, Section 4]). The following theorems appear in Biane [1, Proof of Theorem 1.1], Goulden and Rattan [7, Theorems 1.3 and 3.3] and Śniady [14, Section 1.3.3].

Theorem 1.5 (Biane).

$$\Sigma_{k,0} = R_{k+1}.$$

That is, there is only one term of weight $k+1$ in Σ_k and it is R_{k+1}.

Theorem 1.6 (Goulden-R, Śniady).

$$\Sigma_{k,2} = \frac{1}{4}\binom{k+1}{3} C_{k-1}.$$

Consequently, $\Sigma_{k,2}$ is C-positive.

Theorem 1.7 (Goulden-R). *$\Sigma_{k,4}$ is C-positive.*

Finally, the following theorem is a corollary of [7, Theorem 2.1].

Theorem 1.8. *For $k \geq 1$,*

$$\Sigma_{k,2n} = \sum_{\substack{i_1, i_2, \ldots, i_{2n-1} \geq 0 \\ i_1 + i_2 + \cdots + i_{2n-1} = k+1-2n}} \gamma_{i_1, i_2, \ldots, i_{2n-1}} C_{i_1} \cdot C_{i_2} \cdots C_{i_{2n-1}}$$

where the C_t are given in (7) and the γ's are rational. In particular, $\Sigma_{k,2n}$ is C-positive (and, consequently, R-positive) if all $\gamma_{i_1, i_2, \ldots, i_{2n-1}}$ are positive.

2 Stanley's Polynomials for Rectangular Shapes

In this section we study a specific two variable case of Stanley's results as they have a particularly beautiful form.

We begin with the normalized character $\widehat{\chi}_\omega$ when ω has the rectangular shape of p parts, all equal to q. In Section 1.1, we denoted this shape with $p \times q$. Further, in Section 1.1 we gave the central theorem in this case as Theorem 1.3. This result can be written in terms of the *connection coefficients* of the symmetric group; Theorem 1.3 then becomes

$$\widehat{\chi}_{p\times q}(\mu\, 1^{n-k}) = (-1)^k \sum_{u,\nu \vdash k} c_{u,\nu}^{\mu} p^{\ell(u)} (-q)^{\ell(\nu)}.$$

Here the $c_{u,\nu}^{\mu}$ are defined as the structure constants of the central elements K_u of the group algebra of $\mathfrak{S}_n$; that is,

$$K_u K_\nu = \sum_{\mu} c_{u,\nu}^{\mu} K_\mu.$$

Stanley's proof of this involves a combination of results; Stanley uses results about certain tableaux, the Murnaghan-Nakayama rule, and the following symmetric function identity

$$\sum_{\omega\vdash k} H_\omega\ s_\omega(x)s_\omega(y)s_\omega(z) = \sum_{\omega\vdash k} p_\omega(x)p_\omega(y)p_\omega(z),$$

which appears in Hanlon et al. [8] (here, $s_\omega(x)$ and $p_\omega(x)$ are the Schur symmetric function and power sum symmetric function, respectively). Here, we present an original proof with the aim of making the result more transparent and, in addition, of showing more connections between what are known as *shift symmetric functions* and the normalized character $\widehat{\chi}_{p\times q}$ (we shall see that there is already a known relationship between these objects). Sections 2.1 gives the necessary background on shift symmetric functions for this proof.

2.1 A Brief Account of Shift Symmetric Functions

In Okounkov and Olshanski [12], the authors define *shift Schur polynomials* as

$$s^*_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det\left((x_i + n - i)_{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left((x_i)_{n-j}\right)_{1\le i,j\le n}}.$$

The *shift Schur functions* denoted by $s^*_\lambda \in \Lambda^*$ are defined as the inverse limit of the sequence $(s^*_\lambda(x_1, x_2, \ldots, x_n))_{n\ge1}$. Further, one can define the *p-sharp* shift symmetric functions $p^\sharp_\mu$; they are

$$p^\sharp_\mu = \sum_{\rho\vdash k} \chi_\rho(\mu) s^*_\rho$$

(see Okounkov and Olshanski [12, Section 1] for more details).

The following result connects shift symmetric functions and the scaled characters $\widehat{\chi}$, and can be found in Okounkov and Olshanski [12, (15.21)].

Theorem 2.1 (Okounkov, Olshanski). *Suppose that $\mu \vdash k$ and $\lambda \vdash n$. Then*

$$p^\sharp_\mu(\lambda) = \widehat{\chi}_\lambda(\mu\, 1^{n-k}),$$

where $p^\sharp(\lambda)$ is the substitution $x_i = \lambda_i$ for $1 \le i \le \ell(\lambda)$ and $x_i = 0$ for $i > \ell(\lambda)$.

The following theorem gives a combinatorial interpretation to shift Schur functions; it is also found in Okounkov and Olshanski [12, Theorem 11.1]. For any shape μ, a *reverse tableau of shape* μ is a function T : boxes of $\mu \mapsto \mathbb{P}$, where $\mathbb{P}$ is the set of positive integers, such that T is weakly decreasing along the rows of μ and strongly decreasing along the columns of μ. We denote by $\mathrm{RTab}(\mu)$ the set of reverse tableau of shape μ.

Theorem 2.2 (Okounkov, Olshanski). *For $\lambda \in \mathcal{P}$,*

$$s^*_\lambda = \sum_{T\in\mathrm{RTab}(\mu)} \prod_{u\in\mu} (x_{T(u)} - c(u)),$$

where $T(u)$ is the value assigned to the box u by the tableau T and, again, $c(u)$ is the content of the box u.

2.2 Proof of Theorem 1.3

We are now ready to give a proof of Theorem 1.3.

Proof of Theorem 1.3. As a first step to this proof, for a partition $\lambda \vdash k$ we evaluate $s^*_\lambda(x_1, x_2, \ldots, x_p)$ with $x_i = q$ for $1 \le i \le p$; that is, we compute the evaluation $s^*_\lambda(p \times q)$. Using Theorem 2.2 we obtain

$$\begin{aligned}
s^*_\lambda(p \times q) &= \sum_{T \in \mathrm{RTab}(\lambda)} \prod_{u \in \lambda} (x_{T(u)} - c(u)) \Big|_{(x_1,\ldots,x_p)=(q,\ldots,q)} \\
&= \sum_{T \in \mathrm{RTab}(\lambda)} \prod_{u \in \lambda} (q - c(u)) \\
&= (-1)^k \prod_{u \in \lambda} (-q + c(u)) \sum_{T \in \mathrm{RTab}(\lambda)} 1 \Big|_{(x_1,\ldots,x_p)=(q,\ldots,q)} . \qquad (10)
\end{aligned}$$

The number of $\mathrm{RTab}(\lambda)$ is clearly the number of *semi-standard Young tableaux* (see Stanley [15, page 309]) of shape λ filled with only numbers $1, 2, \ldots, p$, which is $s_\lambda(\mathbf{1^p})$ from the combinatorial definition of Schur functions. Thus, from (10) above and the well known specialization of the Schur functions

$$s_\lambda(\mathbf{1^p}) = \frac{\prod_{u \in \lambda}(p + c(u))}{H_\lambda},$$

where $s_\lambda(\mathbf{1^p})$ is obtained by setting $x_i = 1$ for all $1 \le i \le p$ and $x_i = 0$ for all $i > p$ in the Schur function $s_\lambda(\mathbf{x})$.

$$\begin{aligned}
s^*_\lambda(p \times q) &= (-1)^k \prod_{u \in \lambda} (-q + c(u))\, s_\lambda(\mathbf{1^p}) \\
&= \frac{(-1)^k}{H_\lambda} \prod_{u \in \lambda} (-q + c(u))(p + c(u)).
\end{aligned}$$

Therefore, from Theorem 2.1 and (10) we have

$$\begin{aligned}
\widehat{\chi}_{p \times q}(\mu\, 1^{n-k}) &= \sum_{\lambda \vdash k} \chi_\lambda(\mu) s^*_\lambda(p \times q) \\
&= (-1)^k \sum_{\lambda \vdash k} \frac{\chi_\lambda(\mu)}{H_\lambda} \prod_{u \in \lambda} (p + c(u))(-q + c(u)) \\
&= (-1)^k \sum_{\alpha, \beta, \lambda \vdash k} \frac{\chi_\lambda(\mu)}{H_\lambda} \frac{|C_\alpha|}{f^\lambda} \chi_\lambda(\alpha) p^{\ell(\alpha)} \frac{|C_\beta|}{f^\lambda} \chi_\lambda(\beta) (-q)^{\ell(\beta)} \\
&= (-1)^k \sum_{\alpha, \beta, \vdash k} p^{\ell(\alpha)} (-q)^{\ell(\beta)} \frac{|C_\alpha||C_\beta|}{k!} \sum_{\lambda \vdash k} \frac{1}{f^\lambda} \chi_\lambda(\alpha)\chi_\lambda(\beta)\chi_\lambda(\mu) \\
&= (-1)^k \sum_{\alpha, \beta \vdash k} p^{\ell(\alpha)} (-q)^{\ell(\beta)} c^\mu_{\alpha,\beta},
\end{aligned}$$

where the third equality follows from the well known identity

$$\prod_{u \in \lambda} (x + c(u)) = \sum_{\beta \vdash n} \frac{|C_\beta|}{f^\lambda} \chi_\lambda(\beta) x^{\ell(\beta)}. \qquad (11)$$

and the last equality follows from the well known identity

$$[K_\mu] K_\alpha K_\beta = \frac{|C_\alpha||C_\beta|}{k!} \sum_{\lambda \vdash k} \frac{1}{f^\lambda} \chi_\lambda(\alpha)\chi_\lambda(\beta)\chi_\lambda(\mu)$$

(see Jackson [9, Lemma 2.4]). This completes the proof. □

3 Generalizations to Non-Rectangular Shapes

We now deal with the case of a general shape $\mathbf{p} \times \mathbf{q}$ but, as mentioned in Section 1.1, we are concerned with the series $F_\mu(\mathbf{p};\ \mathbf{q})$ when μ has a single part; that is, we are only concerned with the series $F_k(\mathbf{p};\ \mathbf{q})$. The expressions $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ for $k = 1, 2, 3, 4$ and $m = 2$ are given in (12). These data also appear in Stanley [16, page 8].

$$\begin{aligned}
-F_1(a,p;\ -b,-q) &= ab + pq, \\
F_2(a,p;\ -b,-q) &= a^2b + ab^2 + 2\,apq + p^2q + pq^2, \\
-F_3(a,p;\ -b,-q) &= a^3b + 3\,a^2b^2 + 3\,a^2pq + ab^3 + 3\,abpq + 3\,ap^2q \\
&\quad + 3\,apq^2 + p^3q + 3\,p^2q^2 + pq^3 + ab + pq, \\
F_4(a,p;\ -b,-q) &= a^4b + 6\,a^3b^2 + 4\,a^3pq + 6\,a^2b^3 + 12\,a^2bpq \\
&\quad + 6\,a^2p^2q + 6\,a^2pq^2 + ab^4 + 4\,ab^2pq + 4\,abp^2q \\
&\quad + 4\,abpq^2 + 4\,ap^3q + 14\,ap^2q^2 + 4\,apq^3 + p^4q \\
&\quad + 6\,p^3q^2 + 6\,p^2q^3 + pq^4 + 5\,a^2b + 5\,ab^2 + 10\,apq + 5\,p^2q \\
&\quad + 5\,pq^2.
\end{aligned} \tag{12}$$

Stanley mentions that the terms of highest degree in $F_k(\mathbf{p};\ \mathbf{q})$, *i.e.* the terms of degree $k+1$, have a particularly nice expression. Keeping with Stanley's notation, let $G_k(\mathbf{p};\ \mathbf{q})$ be the terms of highest degree in $F_k(\mathbf{p};\ \mathbf{q})$. We have the following expression for the generating series of $G_k(\mathbf{p};\ \mathbf{q})$, which we call $G_{\mathbf{p};\,\mathbf{q}}(z)$ This theorem appears, with proof, in [16, Proposition 2].

Theorem 3.1 (Stanley). *The generating series for* $G_k(\mathbf{p};\ \mathbf{q})$ *is*

$$G_{\mathbf{p};\,\mathbf{q}}(z) = 1 + \sum_{i\geq 1} G_{i-1}(\mathbf{p};\ \mathbf{q})z^i = \frac{z}{\left(\dfrac{z\displaystyle\prod_{i=1}^{m}\left(1 - \left(q_i + \sum_{j=i+1}^{m} p_j\right)z\right)}{\displaystyle\prod_{i=1}^{m}\left(1 - \left(q_i + \sum_{j=i}^{m} p_j\right)z\right)} \right)^{\langle -1\rangle}}. \tag{13}$$

Of course, $\mathbf{p},\mathbf{q}$-positivity of $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ would imply that $(-1)^k G_k(\mathbf{p};\ -\mathbf{q})$ is also $\mathbf{p},\mathbf{q}$-positive. Stanley does not prove $\mathbf{p},\mathbf{q}$-positivity for the latter series in [16] but states that S. Elizalde has proven this in a private communication to him. In fact, Elizalde shows (according to Stanley)

$$\begin{aligned}
(-1)^k G_k(\mathbf{p};\ \mathbf{q}) &= \frac{1}{k} \sum_{i_1+\cdots+i_m+j_1+\cdots+j_m=k+1} \binom{k}{i_1} \left(\!\binom{i_1}{j_1}\!\right) \\
&\prod_{s=2}^{m} \left(\sum_{r=0}^{\min(i_s,j_s)} \binom{k}{r} \left(\!\binom{r}{j_s - r}\!\right) \binom{k - r - i_i - \cdots - i_{s-1} - j_1 - \cdots - j_{s-1}}{i_s - r} \right) \\
&\qquad\qquad \cdot p_1^{i_1} \cdots p_m^{i_m} q_1^{i_1} \cdots q_m^{i_m},
\end{aligned} \tag{14}$$

where $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. However, as far as this author can see, no proof exists in the literature.

In the next sections we give partial answers to the positivity questions concerning $(-1)^k F(\mathbf{p};\ -\mathbf{q})$. As alluded to in the Section 1, we use Kerov's polynomials to answer these questions.

4 Applying Kerov Polynomials to Stanley's Polynomials

Note that both Kerov's polynomials, along with (3), and (1) give expressions for the scaled character $\widehat{\chi}_\omega$. Since they hold for any shapes $\mathbf{p}\times\mathbf{q}$, we can conclude that they give the same expression for $\widehat{\chi}_\omega$. Thus, we will use (3) and (5) to obtain results about Stanley's polynomials. More specifically, using (3) we obtain the R_i in Kerov's polynomials for the general shape $\mathbf{p}\times\mathbf{q}$; we then use the R_i along with Theorems 1.5, 1.6 and 1.7 to give some positivity results for Stanley's polynomials. The main theorem needed to give our positivity results is given in Theorem 4.3 of Section 4.2; also in Section 4.2 we show, using Theorems 4.3 and 1.5, that the terms of highest degree of Stanley's polynomials are positive. In Section 4.3 we use Theorem 4.3 and Theorems 1.6 and 1.7 to prove the positivity of the terms of degree $k-1$ and $k-3$ in $F_k(\mathbf{p};\ \mathbf{q})$. Finally, we end the paper by showing in Theorem 4.8 that C-positivity for Kerov's polynomials implies $\mathbf{p},\mathbf{q}$-positivity for Stanley's polynomials.

4.1 The Series H for the Shape $\mathbf{p}\times\mathbf{q}$

We now compute what the series H in (2) must be for the shape $\mathbf{p}\times\mathbf{q}$. For the shape $\mathbf{p}\times\mathbf{q}$, it is not difficult to see that its interlacing sequence of maxima and minima is

$$x_1 = q_1,\quad y_1 = q_1 - p_1,\quad x_2 = q_2 - p_1,\quad y_2 = q_2 - p_1 - p_2,\quad x_3 = q_3 - p_1 - p_2,$$

$$y_3 = q_3 - p_1 - p_2 - p_3,\quad \ldots,\quad x_{m-1} = q_m - \sum_{i=1}^{m-1} p_i,\quad y_m = q_m - \sum_{i=1}^{m} p_i,\quad x_m = -\sum_{i=1}^{m} p_i.$$

From (2), we have

$$\begin{aligned} H_{\mathbf{p}\times\mathbf{q}}(1/z) &= \frac{z\left(1-(q_1-p_1)z\right)\left(1-(q_2-(p_1+p_2))z\right)\cdots\left(1-\left(q_m-\sum_{i=1}^{m}p_i\right)z\right)}{\left(1-q_1z\right)\left(1-(q_2-p_1)z\right)\cdots\left(1-\left(q_m-\sum_{i=1}^{m-1}p_i\right)z\right)\left(1+\sum_{i=1}^{m}p_i\right)} \\ &= \frac{z\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}{\left(1+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)}, \end{aligned} \tag{15}$$

and we obtain from (4)

$$R_{\mathbf{p}\times\mathbf{q}}(z) = \frac{z}{\left(\dfrac{z\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}{\left(1+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)}\right)^{\langle -1\rangle}}. \tag{16}$$

Alternatively, it follows from Lagrange inversion (see Goulden and Jackson [6, Section 2] or Stanley [15, Theorem 5.4.2]) that if

$$\begin{aligned}\phi_{\mathbf{p}\times\mathbf{q}}(z) &= \frac{z}{H_{\mathbf{p}\times\mathbf{q}}(1/z)} \\ &= \frac{\left(1+\sum_{j=1}^{m} p_j z\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1} p_j\right) z\right)}{\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i} p_j\right) z\right)},\end{aligned} \tag{17}$$

then

$$\frac{z}{R_{\mathbf{p}\times\mathbf{q}}(z)} = z\,\phi_{\mathbf{p}\times\mathbf{q}}\left(\frac{z}{R_{\mathbf{p}\times\mathbf{q}}(z)}\right). \tag{18}$$

Applying Lagrange inversion, we obtain for $k \geq 2$

$$\begin{aligned}R_k(\mathbf{p}\times\mathbf{q}) &= [z^{k-1}]\frac{R(z)}{z} \\ &= \frac{1}{k-1}[y^{k-2}] - \frac{1}{y^2}\phi_{\mathbf{p}\times\mathbf{q}}^{k-1}(y) \\ &= -\frac{1}{k-1}[y^k]\phi_{\mathbf{p}\times\mathbf{q}}^{k-1}(y).\end{aligned} \tag{19}$$

Of course, substituting $R_i(\mathbf{p}\times\mathbf{q})$ for R_i in Kerov's polynomials will give us the scaled character $\widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(k\ 1^{n-k})$. In fact, doing so produces polynomials in agreement with Stanley's data. We, therefore, can now use Kerov's polynomials to better understand Stanley's character polynomials. It is clear from (15) and (17) that $R_i(\mathbf{p}\times\mathbf{q})$ is a homogeneous polynomial of degree i in the p's and q's. Therefore, since Kerov's polynomial Σ_k is graded with terms of weight $k+1 (\text{mod } 2)$ (see Biane [1, Proof of Theorem 1.1]) in the R_i's, we see that Stanley's character polynomials are also graded with terms of degree $k+1 (\text{mod } 2)$. We state this now as a proposition, for easy reference later.

Proposition 4.1. *Terms of degree i in $F_k(\mathbf{p};\ \mathbf{q})$ are obtained from the terms of weight i in Kerov's polynomials Σ_k with the R_i's evaluated at the shape $\mathbf{p}\times\mathbf{q}$.*

To further reinforce the idea that we are dealing with polynomials, and to make convenient variable substitutions, we depart from the notation used thus far. We shall replace $R_i(\mathbf{p}\times\mathbf{q})$ with $R_i(\mathbf{p};\ \mathbf{q})$ and $R_{\mathbf{p}\times\mathbf{q}}(z)$ with $R_{\mathbf{p};\ \mathbf{q}}(z)$ to emphasize that these objects are polynomials in p's and q's. We do this analogously with $C(z)$, $H_{\mathbf{p}\times\mathbf{q}}(z)$ and $\phi_{\mathbf{p}\times\mathbf{q}}(z)$; that is, the series $\phi_{\mathbf{p};\ \mathbf{q}}(z)$, will denote the series in (17) and $H_{\mathbf{p};\ \mathbf{q}}(z)$ will denote the series in (15). We shall deal with the terms of different weights separately, starting with the terms of highest degree, namely the terms of degree $k+1$.

4.2 Terms of Degree $k+1$

The expression for the terms of highest degree in Stanley's polynomials are given implicitly by $G_{\mathbf{p};\ \mathbf{q}}(z)$ in (13). From Theorem 1.5 and Proposition 4.1, we can obtain a similar formula for the highest degree terms; that is, the terms of highest degree in $F_k(\mathbf{p};\ \mathbf{q})$, which have degree $k+1$, is given by $R_{k+1}(\mathbf{p};\ \mathbf{q})$ and we see that the generating series for the terms of

highest degree is

$$R_{\mathbf{p};\ \mathbf{q}}(z) = \frac{z}{\left(\dfrac{z \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i} p_j \right) z \right)}{\left(1 + \sum_{j=1}^{m} p_j z \right) \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i-1} p_j \right) z \right)} \right)^{\langle -1 \rangle}}. \tag{20}$$

Evidently, the two generating series $R_{\mathbf{p};\ \mathbf{q}}(z)$ and $G_{\mathbf{p};\ \mathbf{q}}(z)$ should be equal; after all they both generate the highest degree terms of $F_k(\mathbf{p};\ \mathbf{q})$, although it is not obvious from (13) and (20) that this is the case. It turns out that $R_{\mathbf{p};\ \mathbf{q}}(z)$ and $G_{\mathbf{p};\ \mathbf{q}}(z)$ are *almost* the same; we state this more precisely in the next proposition.

Proposition 4.2. *The generating series $R_{\mathbf{p};\ \mathbf{q}}(z)$ and $G_{\mathbf{p};\ \mathbf{q}}(z)$ are identical except for the linear terms; more precisely*

$$R_{\mathbf{p};\ \mathbf{q}}(z) = G_{\mathbf{p};\ \mathbf{q}}(z) - \sum_{i=1}^{m} p_i z.$$

Proof. From Lagrange inversion, it suffices to show that $R_{\mathbf{p};\ \mathbf{q}}(z) + \sum_{i=1}^{m} p_i z$ satisfies the same equation as $G_{\mathbf{p};\ \mathbf{q}}(z)$. In this proof, we denote $R_{\mathbf{p};\ \mathbf{q}}(z)$ and $G_{\mathbf{p};\ \mathbf{q}}(z)$ by R and G, respectively, for convenience. From (20) we have

$$\frac{z}{R} = \left(\frac{z \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i} p_j \right) z \right)}{\left(1 + \sum_{j=1}^{m} p_j z \right) \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i-1} p_j \right) z \right)} \right)^{\langle -1 \rangle}.$$

By the definition of compositional inverse we have, from the last expression,

$$\begin{aligned}
z &= \frac{z \prod_{i=1}^{m} \left(R - \left(q_i - \sum_{j=1}^{i} p_j \right) z \right)}{\left(R + \sum_{j=1}^{m} p_j z \right) \prod_{i=1}^{m} \left(R - \left(q_i - \sum_{j=1}^{i-1} p_j \right) z \right)} \\
&= \frac{z \prod_{i=1}^{m} \left(\left(R + \sum_{j=1}^{m} p_j z \right) - \left(q_i + \sum_{j=i+1}^{m} p_j \right) z \right)}{\left(R + \sum_{j=1}^{m} p_j z \right) \prod_{i=1}^{m} \left(\left(R + \sum_{j=1}^{m} p_j z \right) - \left(q_i + \sum_{j=i}^{m} p_j \right) z \right)} \\
&= \frac{\dfrac{z}{\left(R + \sum_{j=1}^{m} p_j z \right)} \prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i+1}^{m} p_j \right) \dfrac{z}{\left(R + \sum_{j=1}^{m} p_j z \right)} \right)}{\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i}^{m} p_j \right) \dfrac{z}{\left(R + \sum_{j=1}^{m} p_j z \right)} \right)}.
\end{aligned}$$

Again, from the definition of compositional inverse, we conclude that

$$\frac{z}{\left(R+\sum_{j=1}^{m} p_j z\right)} = \left(\frac{z \displaystyle\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i+1}^{m} p_j\right) z\right)}{\displaystyle\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i}^{m} p_j\right) z\right)} \right)^{\langle -1 \rangle}.$$

Comparing this expression with (13), the result follows. $\square$

Indeed, using Lagrange inversion we see that the linear terms of $R_{\mathbf{p};\,\mathbf{q}}(z)$ and $G_{\mathbf{p};\,\mathbf{q}}(z)$ are 0 and $\sum_{i=0}^{m} p_i z$, respectively. Furthermore, note that although $R_{\mathbf{p};\,\mathbf{q}}(z)$ and $G_{\mathbf{p};\,\mathbf{q}}(z)$ differ in the linear term, this has no effect on either Kerov's or Stanley's polynomials since $R_1(\mathbf{p};\,\mathbf{q})$ does not appear in Kerov's polynomials, as one can see in (6) (in general, this fact follows from the combinatorial argument given in Biane [1, Proof of Theorem 1.1]).

Through Lagrange inversion, we see that the R_i are written in terms of the series $\phi_{\mathbf{p};\,\mathbf{q}}$ given in (19). We use the notation $\phi_{\mathbf{p};-\mathbf{q}}$, $R_k(\mathbf{p};-\mathbf{q})$ and $G_k(\mathbf{p};-\mathbf{q})$ to denote that we are substituting $-q_i$ for q_i for all i in these series. We have the following compact expression for the series $\phi_{\mathbf{p};-\mathbf{q}}(-z)$.

Theorem 4.3. *For $\mathbf{p} = p_1, p_2, \ldots, p_m$ and $\mathbf{q} = q_1, q_2, \ldots, q_m$, we have*

$$\phi_{\mathbf{p};-\mathbf{q}}(-z) = \prod_{i=1}^{m} \left(1 + \frac{p_i q_i z^2}{(1 - r_{i-1} z)\,(1 - (q_i + r_i) z)}\right).$$

where $r_i = \sum_{j=1}^{i} p_j$.

Proof. We have, from (17),

$$\phi_{\mathbf{p};-\mathbf{q}}(-z) = \frac{(1 - r_m z) \displaystyle\prod_{i=1}^{m} (1 - (q_i + r_{i-1})\, z)}{\displaystyle\prod_{i=1}^{m} (1 - (q_i + r_i)\, z)}.$$

Now set $A_n(z) = 1 - r_n z, F_0 = 1$ and

$$F_n(z) = A_n(z) \frac{\displaystyle\prod_{i=1}^{n} (1 - (q_i + r_{i-1})\, z)}{\displaystyle\prod_{i=1}^{n} (1 - (q_i + r_i)\, z)}. \qquad (21)$$

Note that $\phi_{\mathbf{p};-\mathbf{q}}(-z) = F_m(z)$. Then,

$$\begin{aligned}
F_n(z) &= \frac{F_{n-1}(z)}{A_{n-1}(z)} \frac{1-(q_n+r_{n-1})\,z}{1-(q_n+r_n)\,z} A_n(z) \\
&= \frac{F_{n-1}(z)}{A_{n-1}(z)} \frac{A_{n-1}(z)\left(1-\dfrac{q_n z}{A_{n-1}(z)}\right)}{A_{n-1}(z)\left(1-\dfrac{(q_n+p_n)\,z}{A_{n-1}(z)}\right)} A_{n-1}(z)\left(1-\frac{p_n z}{A_{n-1}(z)}\right) \\
&= F_{n-1}(z) \frac{1-\dfrac{(q_n+p_n)z}{A_{n-1}(z)}+\dfrac{p_n q_n z^2}{A_{n-1}^2(z)}}{1-\dfrac{(q_n+p_n)z}{A_{n-1}(z)}} \\
&= F_{n-1}(z)\left(1+\frac{p_n q_n z^2}{A_{n-1}^2(z)\left(1-\frac{(q_n+p_n)z}{A_{n-1}(z)}\right)}\right) \\
&= F_{n-1}(z)\left(1+\frac{p_n q_n z^2}{A_{n-1}(z)\,(1-(q_n+r_n)z)}\right) \\
&= F_{n-1}(z)\left(1+\frac{p_n q_n z^2}{(1-r_{n-1}z)\,(1-(q_n+r_n)z)}\right). \qquad (22)
\end{aligned}$$

Therefore, from (22) we have

$$\begin{aligned}
\phi_{\mathbf{p};-\mathbf{q}}(-z) &= F_m(z) \\
&= \frac{F_m(z)}{F_0(z)} \\
&= \frac{F_m(z)}{F_{m-1}(z)} \cdot \frac{F_{m-1}(z)}{F_{m-2}(z)} \cdots \frac{F_1(z)}{F_0(z)} \\
&= \prod_{i=1}^{m}\left(1+\frac{p_i q_i z^2}{(1-r_{i-1}z)\,(1-(q_i+r_i)z)}\right). \qquad \square
\end{aligned}$$

Corollary 4.4. *$\phi_{\mathbf{p};-\mathbf{q}}(-z)$ is $\mathbf{p},\mathbf{q}$-positive.*

Proof. Each multiplicand in Theorem 4.3 is $\mathbf{p},\mathbf{q}$-positive, making the product $\mathbf{p},\mathbf{q}$-positive. $\square$

Corollary 4.5. *For all $k \geq 1$, the series in p's and q's $(-1)^k R_{k+1}(\mathbf{p};-\mathbf{q})$ and $(-1)^k G_k(\mathbf{p};-\mathbf{q})$ are $\mathbf{p},\mathbf{q}$-positive. That is, the terms of highest degree in $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ all have positive coefficients.*

Proof. The series $(-1)^k G_k(\mathbf{p};-\mathbf{q})$ consists of by definition the terms of highest degree in $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$, and by Proposition 4.2, $(-1)^k G_k(\mathbf{p};-\mathbf{q}) = (-1)^k R_{k+1}(\mathbf{p};-\mathbf{q})$ are equal for all $k \geq 1$. Thus, it suffices to show that $(-1)^k R_{k+1}(\mathbf{p};-\mathbf{q})$ is $\mathbf{p},\mathbf{q}$-positive for all $k \geq 1$.

By (19) we have

$$\begin{aligned}
(-1)^k R_{k+1}(\mathbf{p};-\mathbf{q}) &= (-1)^k\left(-\frac{1}{k}[y^{k+1}]\ \phi^k_{\mathbf{p};-\mathbf{q}}(y)\right) \\
&= \frac{1}{k}[(-y)^{k+1}]\phi^k_{\mathbf{p};-\mathbf{q}}(y) \\
&= \frac{1}{k}[y^{k+1}]\phi^k_{\mathbf{p};-\mathbf{q}}(-y),
\end{aligned}$$

and the result follows. $\square$

4.3 Terms of Degree $k-1$, $k-3$ and a General Connection Between Kerov's Polynomials and Stanley's Polynomials

In this section we deal with terms of degree $k-1$ and $k-3$ in Stanley's polynomials. We note that in Stanley [16] there are no results concerning terms not of highest degree; Stanley comments only on the series $G_{\mathbf{p};-\mathbf{q}}(z)$, the terms of highest degree in $k+1$. Moreover, we note the complication that $(-1)^k\Sigma_k$ has some negative terms when one evaluates the R_i in terms of the shape $\mathbf{p};\ \mathbf{q}$ and substitutes $-q_i$ for all the q_i. More precisely, consider, for example, Σ_5 given in (6). We see from the comments at the beginning of Section 4 that

$$\begin{aligned}(-1)^5F_5(\mathbf{p};-\mathbf{q}) &= (-1)^5\Sigma_5(\mathbf{p};\ \mathbf{q})|_{\mathbf{q}\to-\mathbf{q}}\\ &= (-1)^5\left(R_6(\mathbf{p};-\mathbf{q})+15R_4(\mathbf{p};-\mathbf{q})+5R_2(\mathbf{p};-\mathbf{q})^2\right.\\ &\qquad\qquad \left.+8R_2(\mathbf{p};-\mathbf{q})\right)\\ &= (-1)^5R_6(\mathbf{p};-\mathbf{q})+15(-1)^3R_4(\mathbf{p};-\mathbf{q})\\ &\qquad\qquad -5((-1)R_2(\mathbf{p};-\mathbf{q}))^2+8(-1)R_2(\mathbf{p};-\mathbf{q}).\end{aligned}$$

Note that all terms are $\mathbf{p},\mathbf{q}$-positive except for the term $-5((-1)R_2(\mathbf{p};-\mathbf{q}))^2$. Thus, $\mathbf{p},\mathbf{q}$-positivity would not immediately follow from R-positivity of Kerov's polynomials. For the terms of degree $k-1$ and $k-3$, however, we can use Theorems 1.6 and 1.7. We begin with the following theorem.

Theorem 4.6. *For $k\geq 3$, the terms of degree $k-1$ in $F_k(\mathbf{p};\ \mathbf{q})$ are given by*

$$-\frac{k(k+1)}{24}[y^{k-3}]\phi''_{\mathbf{p};-\mathbf{q}}(y)\phi^{k-1}_{\mathbf{p};-\mathbf{q}}(y).$$

Proof. From Proposition 4.1 and Theorem 1.6, the terms of degree $k-1$ in $F_k(\mathbf{p};\ \mathbf{q})$ are given by

$$\frac{1}{4}\binom{k+1}{3}C_{k-1}(\mathbf{p};\ \mathbf{q}).$$

Setting $w=z/R_{\mathbf{p};\,\mathbf{q}}(z)$ then from (18) we have

$$z=wR_{\mathbf{p};\,\mathbf{q}}(z),\qquad w=z\phi_{\mathbf{p};\,\mathbf{q}}(w),$$

where $\phi_{\mathbf{p};\,\mathbf{q}}(z)$ is given in (17). Further, from the definition of $C_{\mathbf{p};\,\mathbf{q}}(z)$ we have

$$C_{\mathbf{p};\,\mathbf{q}}(z)=\frac{1}{-z^2\frac{d}{dz}\frac{1}{w}},$$

Thus,

$$z\frac{d}{dz}w=\frac{w}{1-z\phi'_{\mathbf{p};\,\mathbf{q}}(w)},$$

from which we obtain

$$\begin{aligned}C_{\mathbf{p};\,\mathbf{q}}(z) &= \frac{1}{-z^2\frac{d}{dz}\frac{1}{w}}\\ &= \frac{1}{\frac{z^2}{w^2}\frac{d}{dz}w}\\ &= \frac{w}{z}(1-z\phi'_{\mathbf{p};\,\mathbf{q}}(w))\\ &= \phi_{\mathbf{p};\,\mathbf{q}}(w)-w\phi'_{\mathbf{p};\,\mathbf{q}}(w).\end{aligned}$$

Therefore, for all $k \geq 2$, we have by Lagrange inversion that

$$\begin{aligned}
[z^{k-1}]\, C_{\mathbf{p};\,\mathbf{q}}(z) &= [z^{k-1}]\, \phi_{\mathbf{p};\,\mathbf{q}}(w) - [z^{k-1}]\, w\phi'_{\mathbf{p};\,\mathbf{q}}(w) \\
&= \frac{1}{k-1}[y^{k-2}]\, \phi'_{\mathbf{p};\,\mathbf{q}}(y)\phi^{k-1}_{\mathbf{p};\,\mathbf{q}}(y) \\
&\qquad - \frac{1}{k-1}[y^{k-2}]\, \left(\phi'_{\mathbf{p};\,\mathbf{q}}(y) + y\phi''_{\mathbf{p};\,\mathbf{q}}(y)\right)\phi^{k-1}_{\mathbf{p};\,\mathbf{q}}(y) \\
&= -\frac{1}{k-1}[y^{k-3}]\phi''_{\mathbf{p};\,\mathbf{q}}(y)\phi^{k-1}_{\mathbf{p};\,\mathbf{q}}(y),
\end{aligned}$$

and the result follows. □

From Theorem 4.6 we obtain the following positivity result.

Corollary 4.7. *For $k \geq 3$, the terms of degree $k-1$ in $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$ are $\mathbf{p},\mathbf{q}$-positive.*

Proof. The terms of degree $k-1$ in $F_k(\mathbf{p};\, \mathbf{q})$ are given in Theorem 4.6. Therefore, the terms of degree $k-1$ in $(-1)^k F_k(\mathbf{p};\, -\mathbf{q})$ are

$$\begin{aligned}
(-1)^k \frac{1}{4}\binom{k+1}{3} C_{k-1}(\mathbf{p}; -\mathbf{q}) &= -\frac{1}{4}\binom{k+1}{3}[z^{k-1}]\, C_{\mathbf{p};-\mathbf{q}}(-z) \\
&= \frac{k(k+1)}{24}[y^{k-1}](-y)^2 \frac{d^2}{d(-y)^2}\left(\phi_{\mathbf{p};-\mathbf{q}}(-y)\right) \\
&\qquad \cdot \phi^{k-1}_{\mathbf{p};-\mathbf{q}}(-y) \\
&= \frac{k(k+1)}{24}[y^{k-1}]y^2 \left(\frac{d^2}{dy^2}(-1)^2\right)\left(\phi_{\mathbf{p};-\mathbf{q}}(-y)\right) \\
&\qquad \cdot \phi^{k-1}_{\mathbf{p};-\mathbf{q}}(-y) \\
&= \frac{k(k+1)}{24}[y^{k-1}]y^2 \frac{d^2}{dy^2}\left(\phi_{\mathbf{p};-\mathbf{q}}(-y)\right)\phi^{k-1}_{\mathbf{p};-\mathbf{q}}(-y).
\end{aligned}$$

From Theorem 4.4 both $\phi_{\mathbf{p};-\mathbf{q}}(-y)$ and, of course then, $\frac{d^2}{dy^2}\phi_{\mathbf{p};-\mathbf{q}}(-y)$ are $\mathbf{p},\mathbf{q}$-positive, proving the result. □

The following theorem gives a general connection between Kerov's polynomials and Stanley's polynomials.

Theorem 4.8. *If Kerov's polynomials Σ_k are C-positive then Stanley's polynomials $(-1)^k F_k(\mathbf{p};\, -\mathbf{q})$ are $\mathbf{p},\mathbf{q}$-positive.*

Proof. From Proposition 4.1 the terms of degree i in Stanley's polynomials are obtained from the terms of weight i in Kerov's polynomials. From Theorem 1.8 the terms of degree $k+1-2n$ in Stanley's polynomials are obtained from

$$\sum_{\substack{i_1,\ldots,i_{2n-1}\geq 0 \\ i_1+\cdots+i_{2n-1}=k+1-2n}} \gamma_{i_1,\ldots,i_{2n-1}} C_{i_1}(\mathbf{p};\, \mathbf{q})\cdots C_{i_{2n-1}}(\mathbf{p};\, \mathbf{q}).$$

Thus, the terms of degree $k+1-2n$ in $(-1)^k F_k(\mathbf{p};\, -\mathbf{q})$ are given by

$$\sum_{\substack{i_1,\ldots,i_{2n-1}\geq 0 \\ i_1+\cdots+i_{2n-1}=k+1-2n}} \gamma_{i_1,\ldots,i_{2n-1}} \left((-1)^{i_1-1}C_{i_1}(\mathbf{p};\, -\mathbf{q})\right)\cdots\left((-1)^{i_{2n-1}-1}C_{i_{2n-1}}(\mathbf{p};\, -\mathbf{q})\right).$$

From the proof of Corollary 4.7, each $(-1)^{j-1}C_j(\mathbf{p};\, -\mathbf{q})$ is $\mathbf{p},\mathbf{q}$-positive, and the result follows. □

Corollary 4.9. *For $k \geq 5$, the terms of degree $k-3$ in $(-1)^k F(\mathbf{p}; -\mathbf{q})$ are $\mathbf{p},\mathbf{q}$-positive.*

Proof. Follows directly from Theorems 1.7 and 4.8. □

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